**CHAPTER 10:**

CUBIC COORDINATES

10.1 REVIEW OF CUBIC COORDINATES

In chapter 1, we defined a coordinate system (x,y,z) to identify locations in 3D by:

- $x =$ units left/right of (0, 0, 0) (+ is right, - is left).
- $y =$ units forward/backward of (0, 0, 0) (+ is forward, - is backward).
- $z =$ units up/down of (0, 0, 0) (+ is up, - is down).

If we consider a the set of points satisfying $a \le x \le b, \ c \le y \le d$ and $e \le z \le f$ where *a, b, c, d, e* and *f* are all constants, this solid will be a cube. For Example, the set of points satisfying $1 \le x \le 3, 2 \le y \le 4$ and $1 \le z \le 5$ is the cube between the two plane segments shown below. For this reason, the coordinates in this system are generally referred to as cubic coordinates.
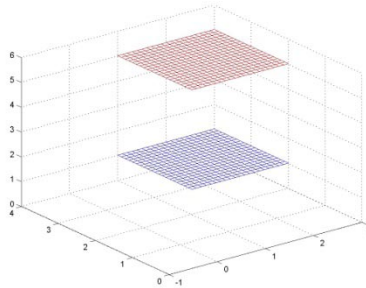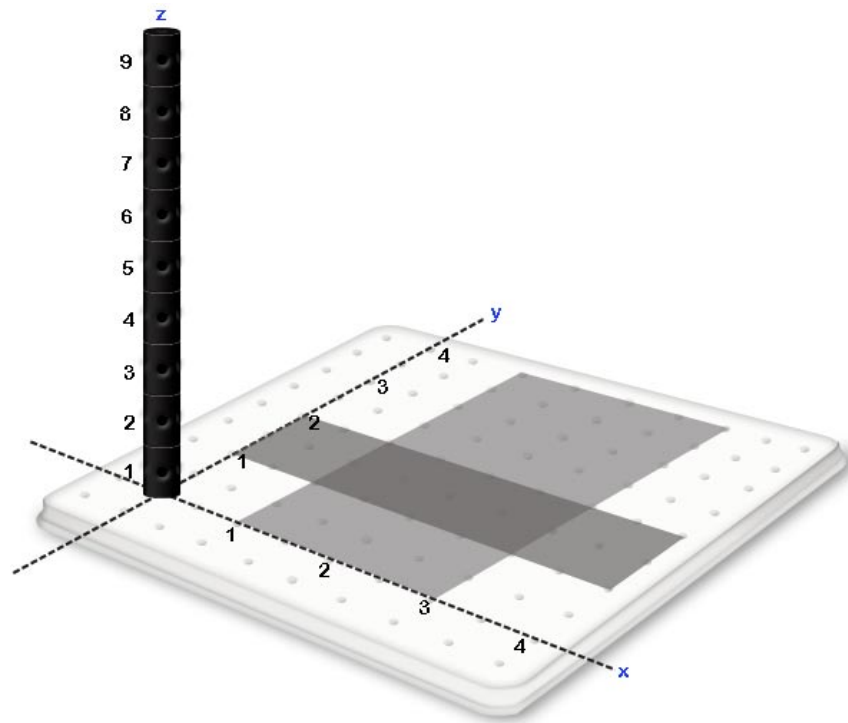

10.2 IDENTIFYING VOLUMES ASSOCIATED WITH TRIPLE INTEGRALS IN CUBIC COORDINATES

Example Exercise 10.2.1: Given the integral $\int_1^3 \int_1^2 \int_1^{2x+3y+1} dz\,dy\,dx$ identify the volume associated with the integral.
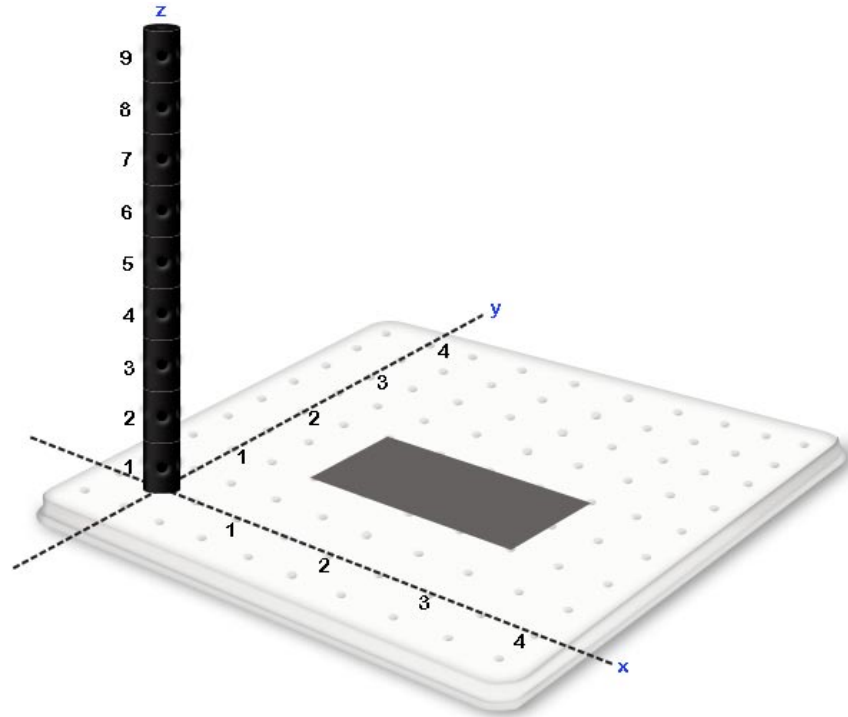
Solution: Working from the outside inward, the first datum from the integral $\int_1^3 dx$ indicating that our volume will contain values of x between $x = 1$ and $x = 3$.
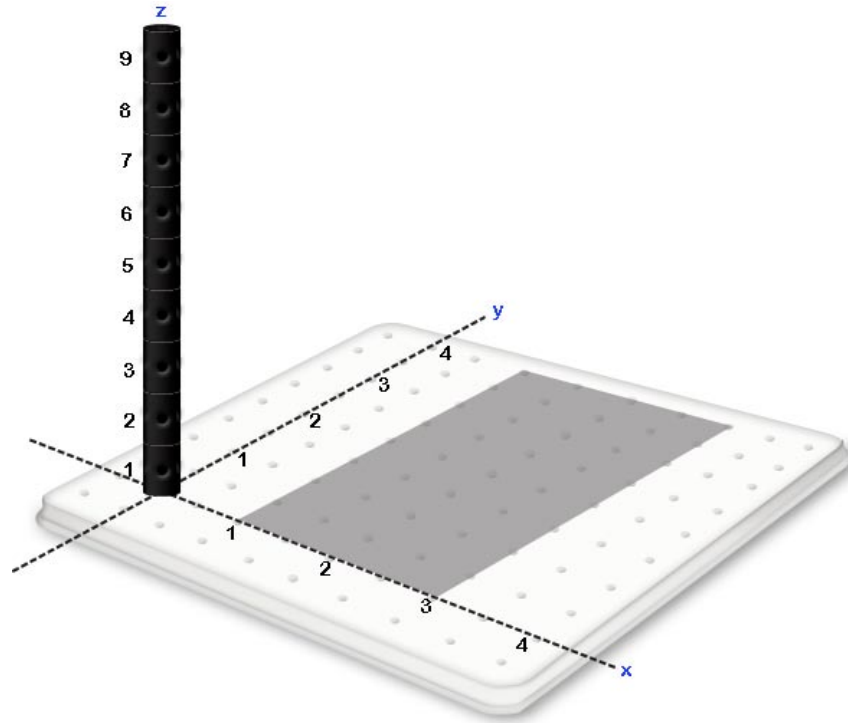
The second datum from the integral is $\int_1^2 dy$ indicating that for every value of x between $x = 1$ and $x = 3$, we will accept values of y that reside between $y = 1$ and $y = 2$.

Together these two data give the following rectangular region in the xy plane.

The final datum in the integral is $\int_1^{2x+y+1} dz$ which indicates that for every (x, y) in the indicated region we wish to include all points above the plane $z = 1$ and below the plane $z = 2x + y + 1$. Taking all data into account, the volume associated with $\int_1^3 \int_1^2 \int_1^{2x+3y+1} dz\,dy\,dx$ is given by the following volume.

z = 2x + y + 1

Example Exercise 10.2.2: Given the integral $\int_1^3 \int_0^x \int_x^{x+3y+2} dz\,dy\,dx$, identify the volume associated with the integral.

Solution: Working from the outside inward, the first datum from the integral $\int_1^3 dx$ indicating that our volume will contain values of x between $x = 1$ and $x = 3$.
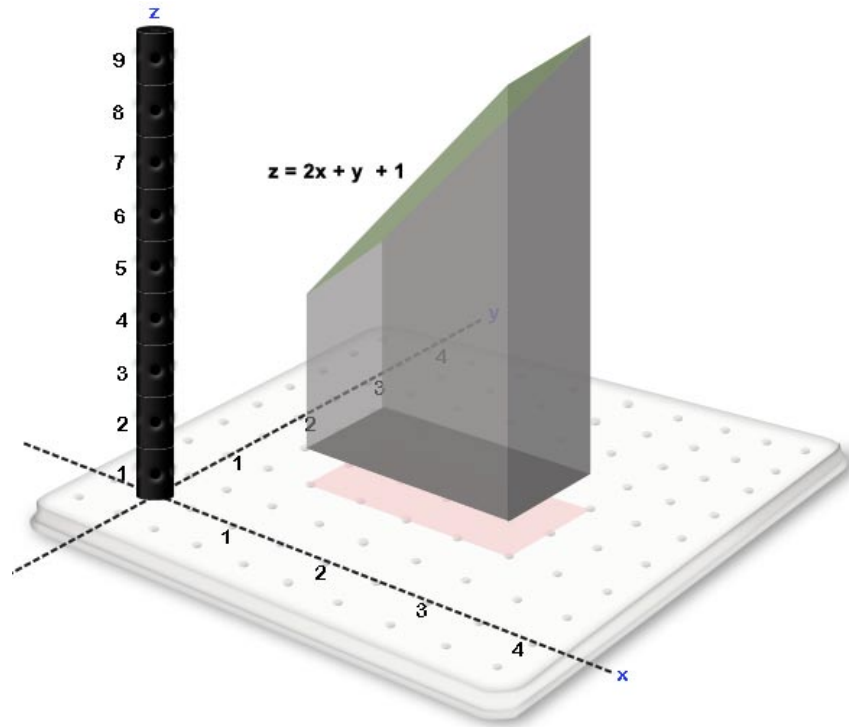
The second datum from the integral is $\int_0^x dy$ indicating that for every value of x between $x = 1$ and $x = 3$, we will accept values of y that reside between $y = 0$ and $y = x$.

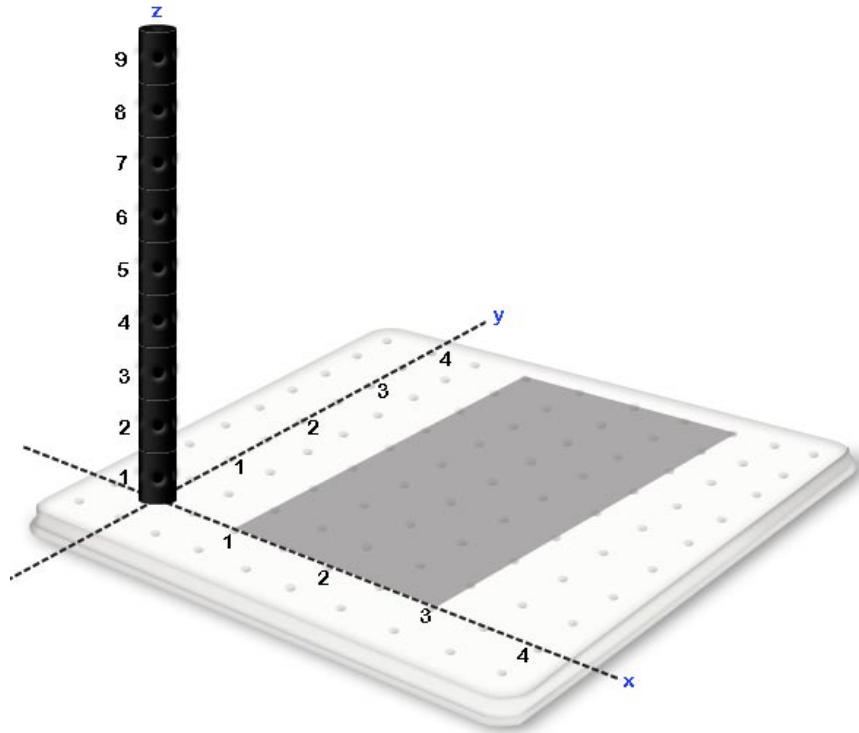These first two data from the following trapezoidal region in the *xy*.

The final datum in the integral is $\int_x^{x+3y+2} dz$ which indicates that for every (x, y) in the indicated region we wish to include all points above the plane $z = x$ and below the plane $z = x + 3y + 2$. Taking all data into account the volume associated with $\int_1^3 \int_0^x \int_x^{x+3y+2} dz\,dy\,dx$ is given by the following volume.
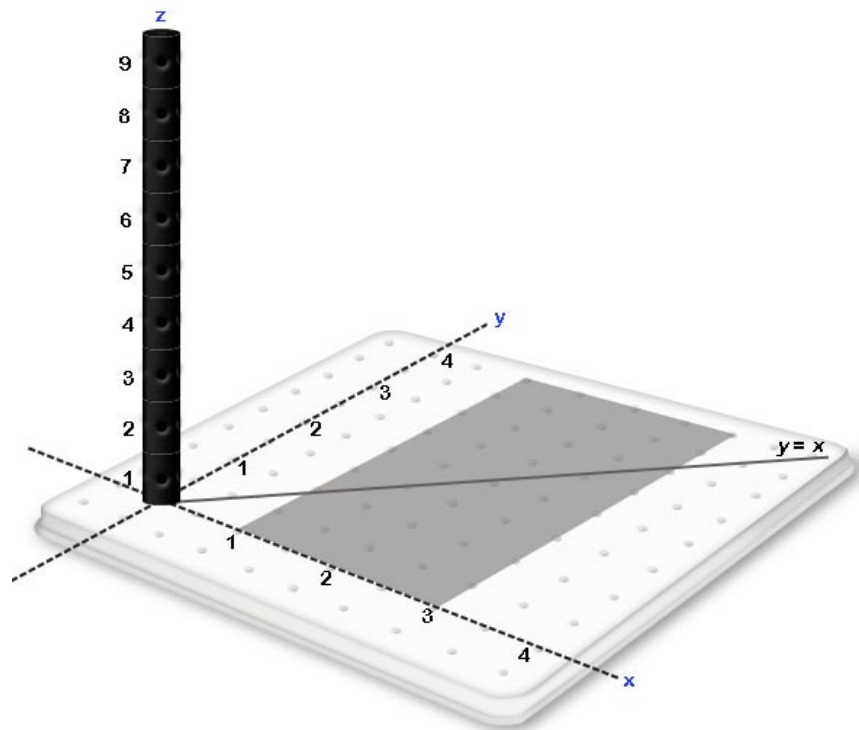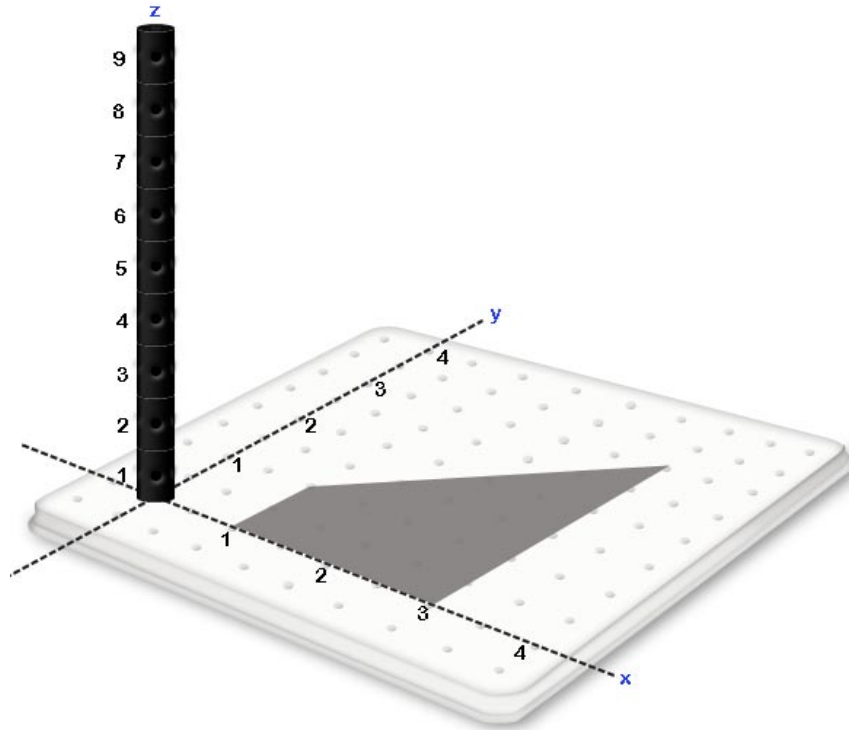
Example Exercise 10.2.3: Given the Integral $\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} \int_{x+y+2}^9 dz\,dx\,dy$, identify the volume associated with the integral.

Solution: Working from the outside inward, the first datum from the integral is $\int_0^4 dy$ indicating that our volume will contain values of y between $y = 0$ and $y = 4$.
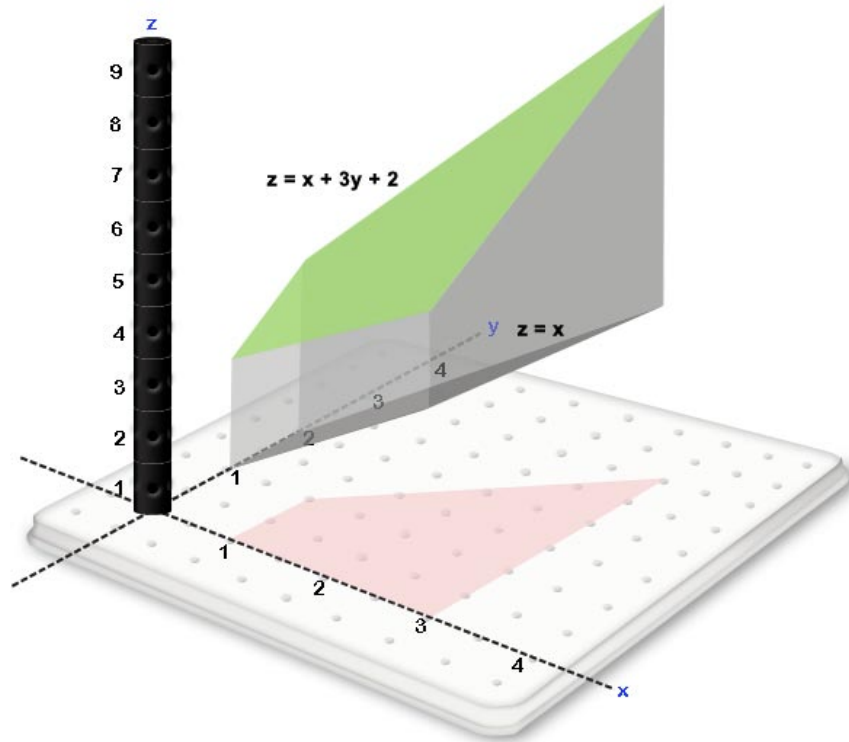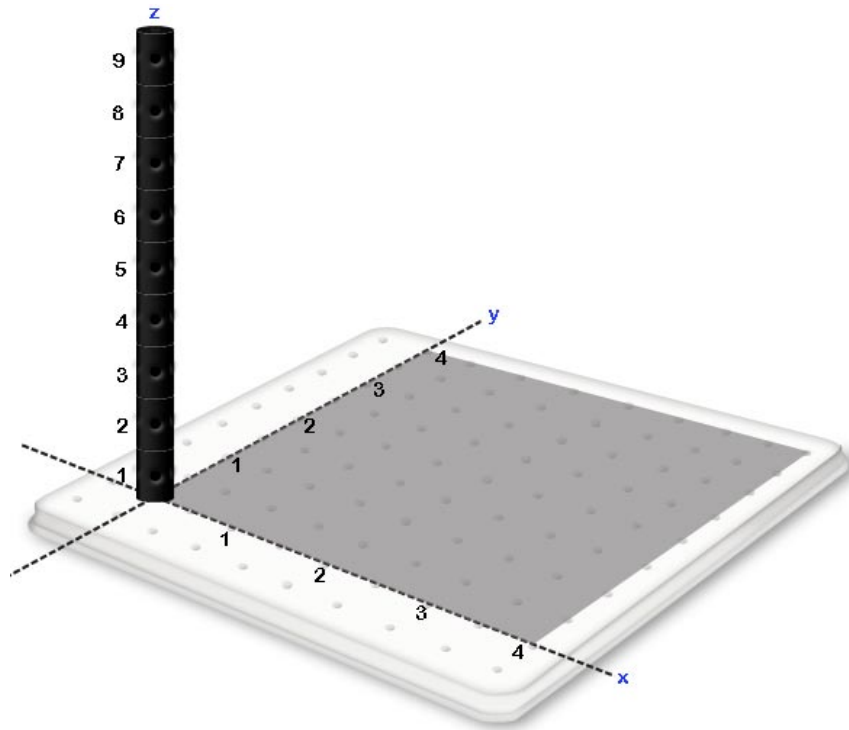
The second datum from the integral is $\int_{\frac{y}{2}}^{\sqrt{y}} dx$ indicating that for every value of y between $y = 1$ and $y = 4$, we will accept values of x that reside between $x = \frac{y}{2}$ and $x = \sqrt{y}$
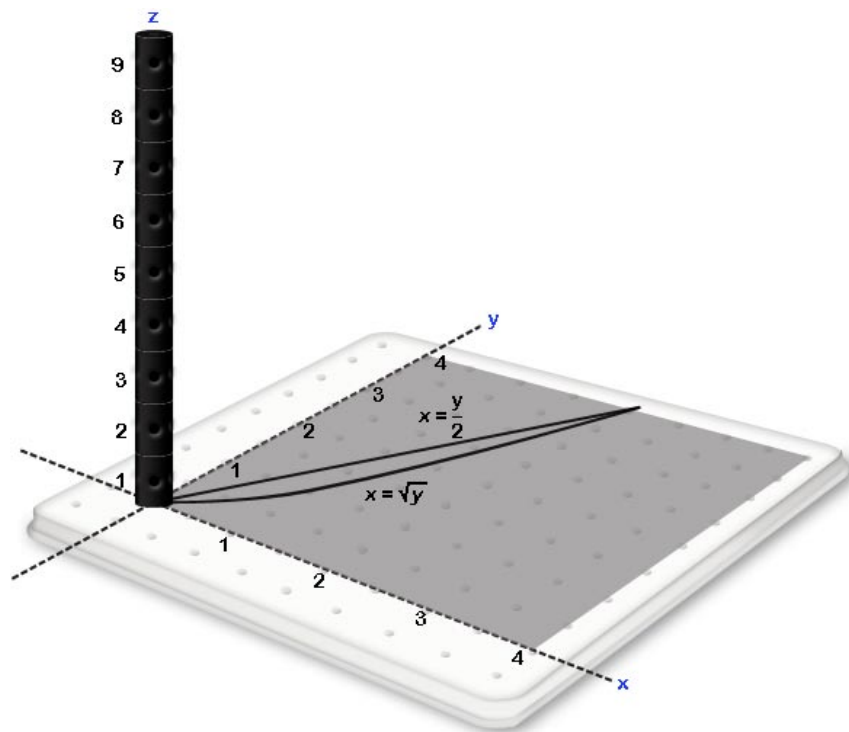
The combination of the two data defined thus far give the following region in the *xy*.
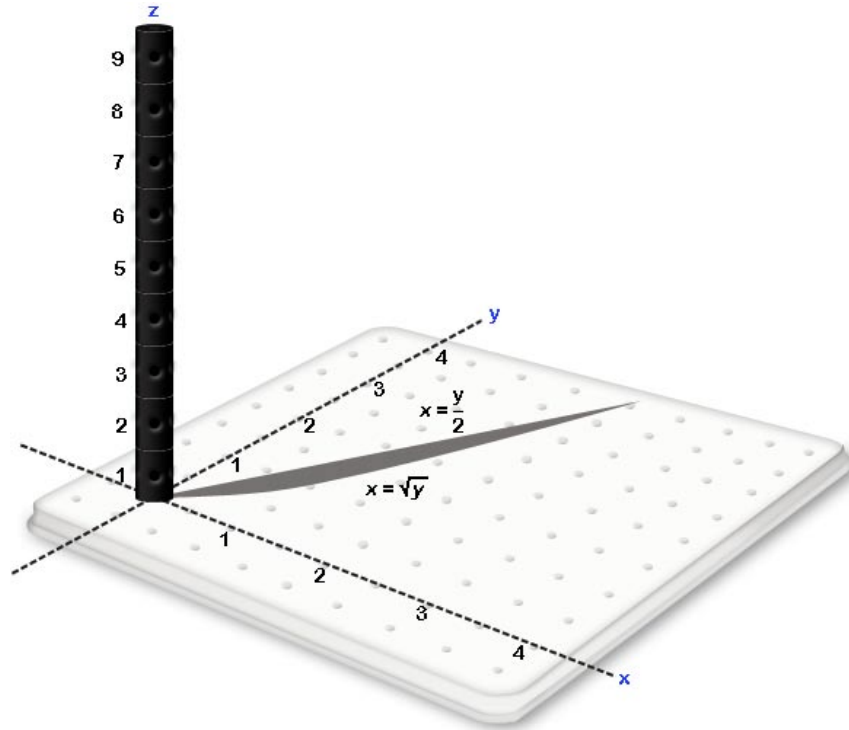
The volume associated with $\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} \int_{x+y+2}^9 dzdxdy$ is given by the volume in this region of the *xy* plane that is above the plane $z = x + y + 2$ and below the plane $z = 9$.
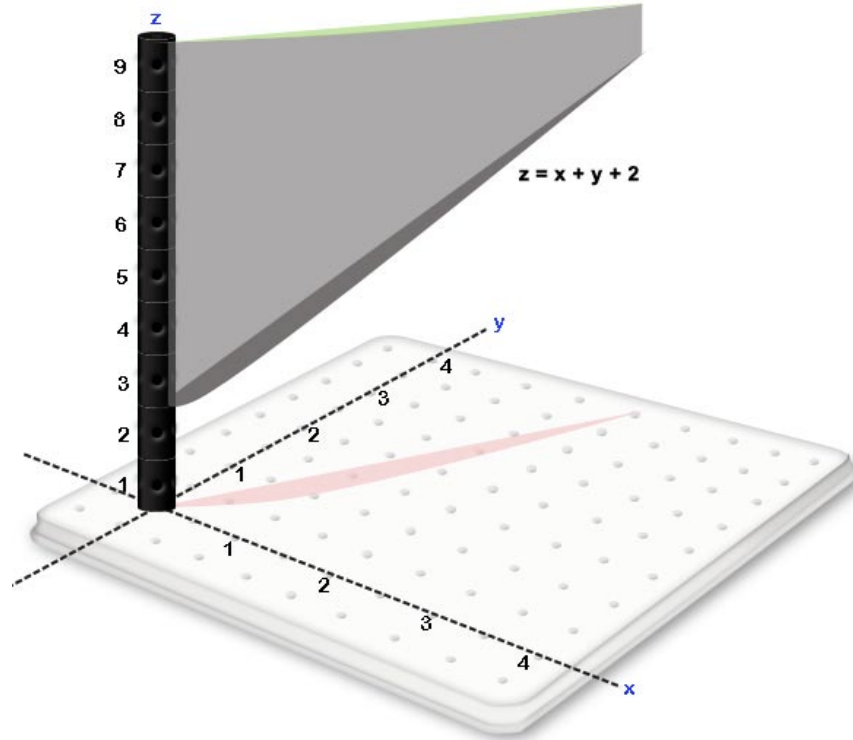
10.3 USING RIEMANN SUMS AND THE FUNDAMENTAL THEOREM TO FIND MASSES ASSOCIATED WITH TRIPLE INTEGRALS

Example Exercise 10.3.1: A solid cube defined by $1 \leq x \leq 3$, $2 \leq y \leq 4$ and $1 \leq z \leq 5$ where x, y and z are measured in meters has a variable density given by the function $density = f(x, y, z) = x + y + z \frac{kg}{m^3}$. Approximate the volume with a Riemann Sum using two divisions in each variable and the maximum value for each variable in each division. Then express the Riemann Sum as a triple summation and apply the fundamental theorem to find the precise mass.

Solution:

Step 1: Divide each variable of the cube into two parts, identifying the maximum value of the variable that will be used in each region.

- $1 \leq x \leq 3$ will be divided into two regions:
  - The first is $1 \leq x \leq 2$ where $x_1 = 2$ will be used.
  - The second is $2 \leq x \leq 3$ where $x_2 = 3$ will be used.

- $2 \leq y \leq 4$ will be divided into two regions:

- The first is $2 \leq y \leq 3$ where $y_1 = 3$ will be used.
- The second is $3 \leq y \leq 4$ where $y_2 = 4$ will be used.

- $1 \leq z \leq 5$ will be divided into two regions:
  - The first is $1 \leq z \leq 3$ where $z_1 = 3$ will be used.
  - The second is $3 \leq z \leq 5$ where $z_2 = 5$ will be used

These divisions are shown in the following diagram.
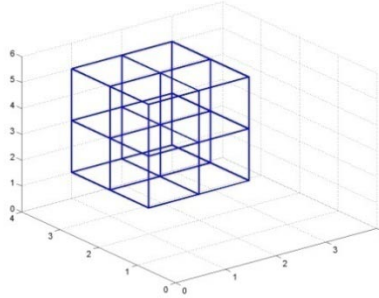
For ease of analysis, the eight divisions shown above can be visualized as a house with four divisions on the first floor and 4 divisions on the second floor as follows:
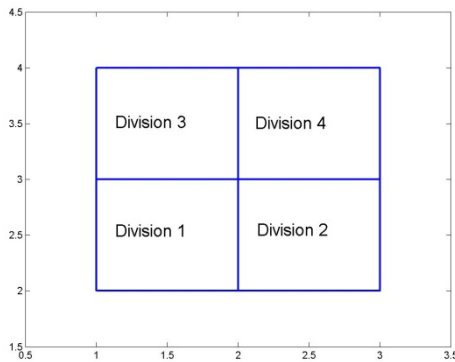
First floor: $1 \leq z \leq 3$
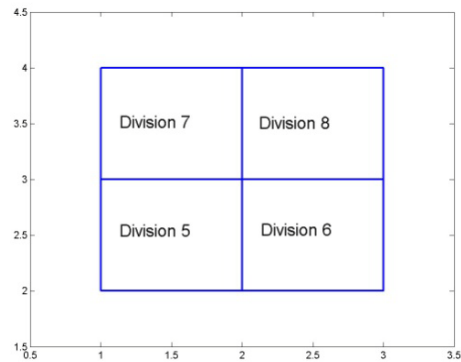
Second floor: $3 \leq z \leq 5$

Step 2: For each division, find the numeric approximation for the length, width, height, volume, density and mass in the division.
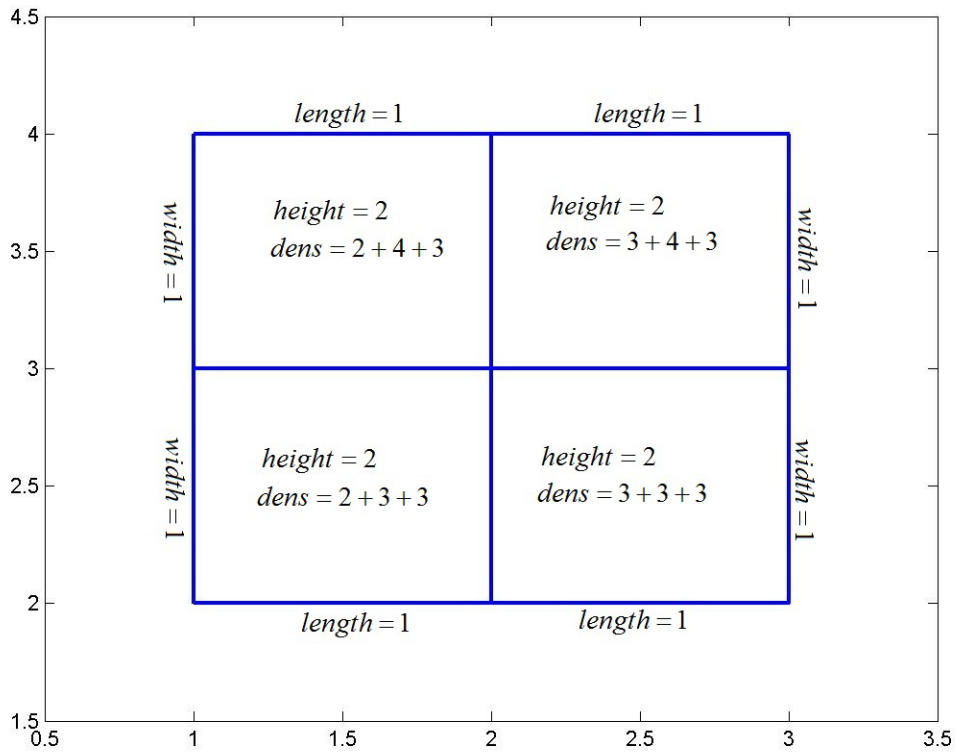
First floor: $1 \leq z \leq 3$
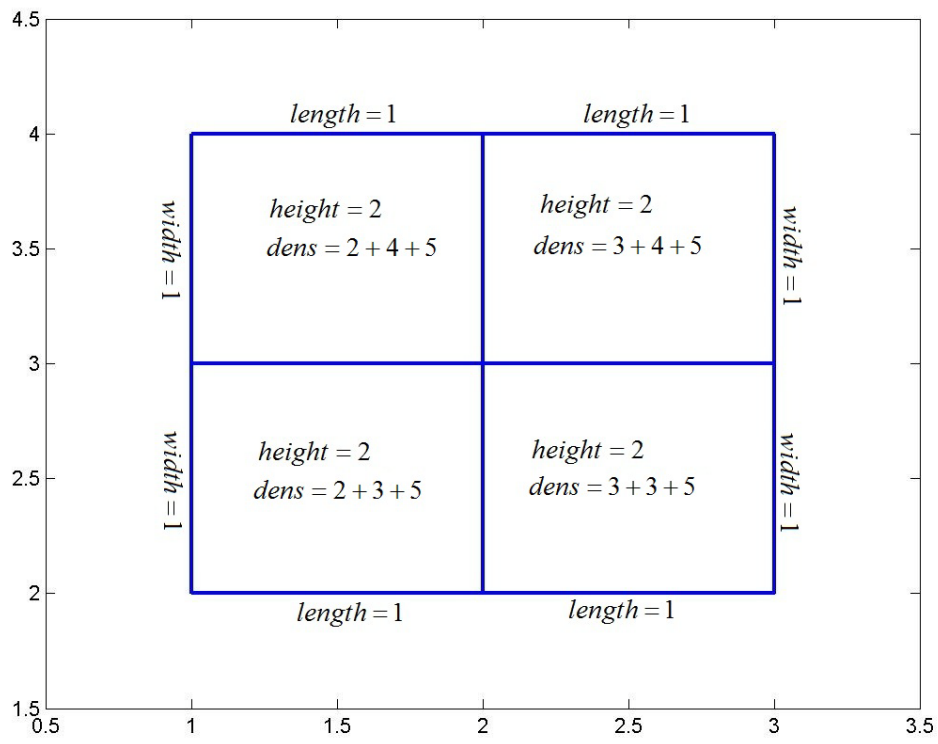
Second floor: $3 \leq z \leq 5$

Organizing the data presented in the preceding diagrams yields the following table:

Division	Length	Width	Height	Volume	Density	Mass
1	1 m	1 m	2 m	$2\ m^3$	$8\ \dfrac{kg}{m^3}$	$16\ kg$
2	1 m	1 m	2 m	$2\ m^3$	$9\ \dfrac{kg}{m^3}$	$18\ kg$
3	1 m	1 m	2 m	$2\ m^3$	$9\ \dfrac{kg}{m^3}$	$18\ kg$
4	1 m	1 m	2 m	$2\ m^3$	$10\ \dfrac{kg}{m^3}$	$20\ kg$
5	1 m	1 m	2 m	$2\ m^3$	$10\ \dfrac{kg}{m^3}$	$20\ kg$
6	1 m	1 m	2 m	$2\ m^3$	$11\ \dfrac{kg}{m^3}$	$22\ kg$
7	1 m	1 m	2 m	$2\ m^3$	$11\ \dfrac{kg}{m^3}$	$22\ kg$
8	1 m	1 m	2 m	$2\ m^3$	$12\ \dfrac{kg}{m^3}$	$24\ kg$

Step 3: Add the masses of the eight divisions to approximate the mass of the entire solid.

$$mass \approx 16\ kg + 18\ kg + 18\ kg + 20\ kg + 20\ kg + 22\ kg + 22\ kg + 24\ kg$$

Step 4: Repeat step 2 using $x_1, x_2, y_1, y_2, z_1, z_2, \Delta x, \Delta y$ and Δz as appropriate:
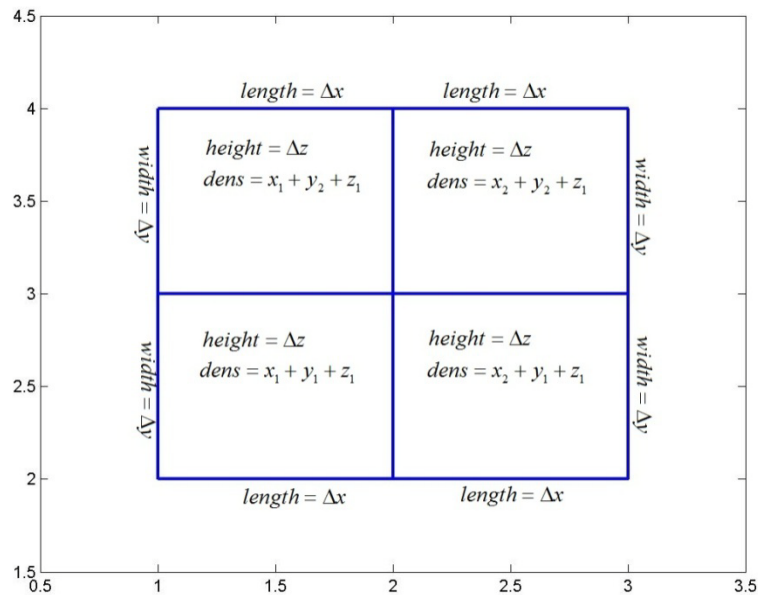
First floor: $1 \le z \le 3$

length = Δx length = Δx

width = Δy

height = Δz
dens = $x_1 + y_2 + z_2$

height = Δz
dens = $x_2 + y_2 + z_2$

width = Δy

width = Δy

height = Δz
dens = $x_1 + y_1 + z_2$

height = Δz
dens = $x_1 + y_2 + z_2$

width = Δy

length = Δx length = Δx

Second floor: $3 \le z \le 5$

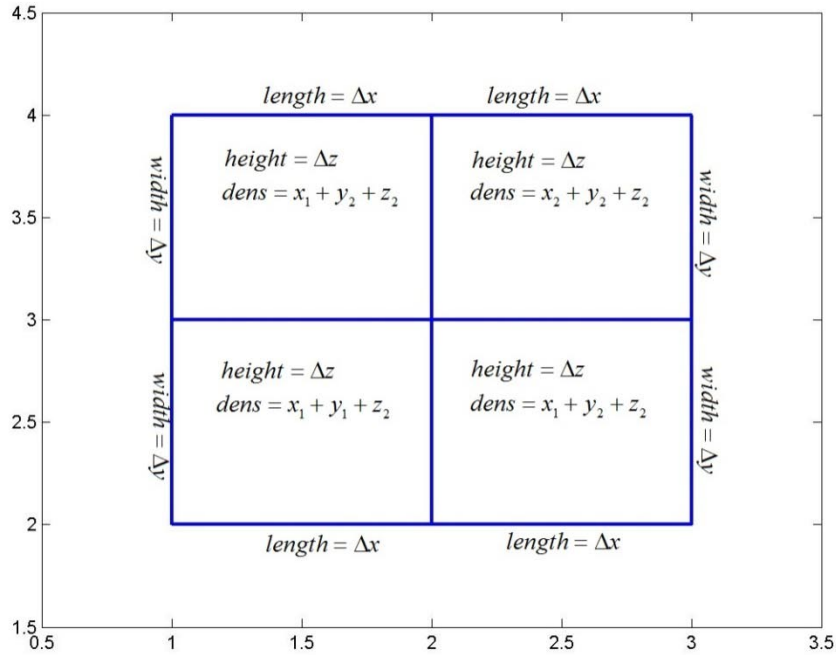Organizing the data presented in the preceding diagrams yields the following table (Note: the units are the same as those in step 2 but are left out for ease of reading):

Division	Length (m)	Width (m)	Height (m)	Volume (m^3)	Density ($\frac{kg}{m^3}$)	Mass (kg)
1	Δx	Δy	Δz	$\Delta x \Delta y \Delta z$	$x_1 + y_1 + z_1$	$(x_1 + y_1 + z_1)\, \Delta x \Delta y \Delta z$
2	Δx	Δy	Δz	$\Delta x \Delta y \Delta z$	$x_2 + y_1 + z_1$	$(x_2 + y_1 + z_1)\, \Delta x \Delta y \Delta z$
3	Δx	Δy	Δz	$\Delta x \Delta y \Delta z$	$x_1 + y_2 + z_1$	$(x_1 + y_2 + z_1)\, \Delta x \Delta y \Delta z$
4	Δx	Δy	Δz	$\Delta x \Delta y \Delta z$	$x_2 + y_2 + z_1$	$(x_2 + y_2 + z_1)\, \Delta x \Delta y \Delta z$
5	Δx	Δy	Δz	$\Delta x \Delta y \Delta z$	$x_1 + y_1 + z_2$	$(x_1 + y_1 + z_2)\, \Delta x \Delta y \Delta z$
6	Δx	Δy	Δz	$\Delta x \Delta y \Delta z$	$x_2 + y_1 + z_2$	$(x_2 + y_1 + z_2)\, \Delta x \Delta y \Delta z$
7	Δx	Δy	Δz	$\Delta x \Delta y \Delta z$	$x_1 + y_2 + z_2$	$(x_1 + y_2 + z_2)\, \Delta x \Delta y \Delta z$
8	Δx	Δy	Δz	$\Delta x \Delta y \Delta z$	$x_2 + y_2 + z_2$	$(x_2 + y_2 + z_2)\, \Delta x \Delta y \Delta z$

Step 5: Repeat step 3 using $x_1, x_2, y_1, y_2, z_1, z_2, \Delta x, \Delta y$ and Δz as appropriate and then express the sum that represents the approximate mass of the solid as a triple summation:

$$mass \approx [(x_1 + y_1 + z_1)\, \Delta x \Delta y \Delta z + (x_2 + y_1 + z_1)\, \Delta x \Delta y \Delta z] \quad + [(x_1 + y_2 + z_1)\, \Delta x \Delta y \Delta z +$$
$$(x_2 + y_2 + z_1)\, \Delta x \Delta y \Delta z] + [(x_1 + y_1 + z_2)\, \Delta x \Delta y \Delta z + (x_2 + y_1 + z_2)\, \Delta x \Delta y \Delta z] +$$
$$[(x_1 + y_2 + z_2)\, \Delta x \Delta y \Delta z + (x_2 + y_2 + z_2)\, \Delta x \Delta y \Delta z]$$

By grouping the [..], this approximation can be expressed as

$$mass \approx \{[\textstyle\sum_{i=1}^{2} x_i + y_1 + z_1)\, \Delta x \Delta y \Delta z] + [\textstyle\sum_{i=1}^{2} x_i + y_2 + z_1)\, \Delta x \Delta y \Delta z]\} + \{[\textstyle\sum_{i=1}^{2} x_i + y_1 +$$
$$z_2)\, \Delta x \Delta y \Delta z] + [\textstyle\sum_{i=1}^{2} x_i + y_2 + z_2)\, \Delta x \Delta y \Delta z]\}$$

By grouping the {..}, this approximation can be expressed as

$$mass \approx \{\sum_{j=1}^{2} \sum_{i=1}^{2} x_i + y_j + z_1) \, \Delta x \Delta y \Delta z \} + \{\sum_{j=1}^{2} \sum_{i=1}^{2} x_i + y_j + z_2) \, \Delta x \Delta y \Delta z \}$$

These last two terms may now be grouped to obtain

$$mass \approx \sum_{k=1}^{2} \sum_{j=1}^{2} \sum_{i=1}^{2} (x_i + y_j + z_k) \, \Delta x \Delta y \Delta z$$

Step 6: Recognizing that this approximation becomes precise as $\Delta x \to 0, \Delta y \to 0$ and $\Delta z \to 0$, apply the fundamental theorem to obtain the precise mass of the solid.

$$mass = \lim_{\Delta z \to 0} \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \sum_{k=1}^{2} \sum_{j=1}^{2} \sum_{i=1}^{2} (x_i + y_j + z_k) \, \Delta x \Delta y \Delta z = \int_{1}^{5} \int_{2}^{4} \int_{1}^{3} (x + y + z) dx dy dz$$

10.4 USING INTEGRALS TO FIND VOLUMES BETWEEN SURFACES

Example Exercise 10.4.1: Find the volume between the surfaces $x = 0, y = 0, z = 0$ and $z = 4 - x - y$.

Solution:

Step 1: Visualize each surface individually and then together in 3 space. If you are unable to visualize a surface well, construct them using cross sections and contours with your 3D kit or use the meshgrid and mesh commands in MATLAB.

$x = 0, y = 0$ and $z = 0$	$z = 4 - x - y$	All Surfaces together
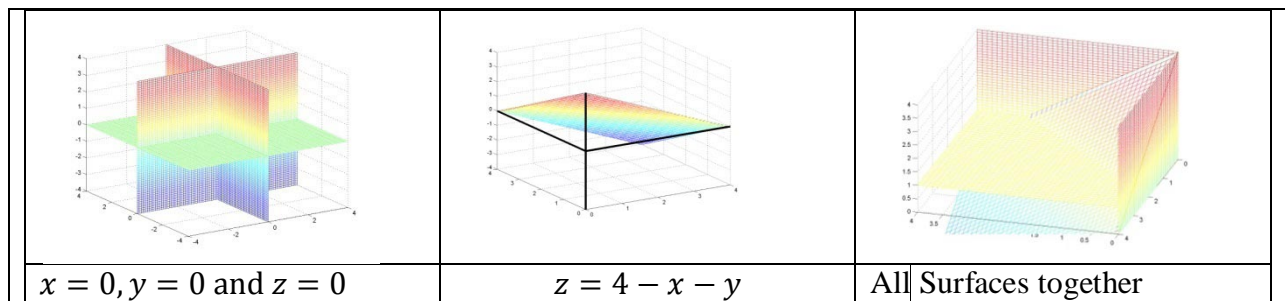
Step 2: Identify the desired solid and find the borders that bound the solid.
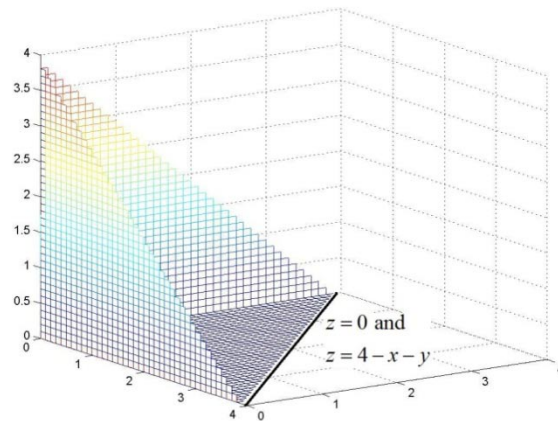
In the above diagram we can see that the solid is bounded below by $z = 0$ and bounded above by $z = 4 - x - y$. Laterally, the solid is bounded its walls. Two of these walls are $x = 0$ and $y = 0$. The third wall is formed by the ceiling which provides both a top and a side to the solid. We wish to find the set of points in the xy plane over which this solid resides. The boundaries of this region in the xy plane are $x = 0$, $y = 0$ and the line shown in the diagram above where the ceiling of the solid meets the floor of the solid. As the line where the floor meets the ceiling will satisfy both $z = 0$ and $z = 4 - x - y$, with simple substitution and a little algebra we find that at this border
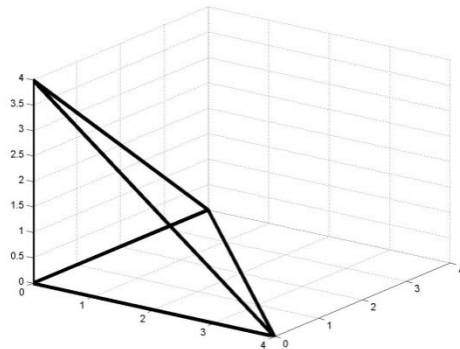
$$4 - x - y = 0$$

$$y = 4 - x$$

Hence the solid is bounded laterally by $x = 0$, $y = 0$ and $y = 4 - x$. It is bounded

below by $z = 0$ and bounded above by $z = 4 - x - y$

Step 3: Define the solid as a region in the xy plane with an associated floor and ceiling.

As we can see in the above diagram, the solid can be viewed as the solid above the triangle with vertices (0,0), (0,4) and (4,0) in the xy plane with floor $z = 0$ and ceiling $z = 4 - x - y$.

Step 4: Define the ranges of x, y and z that will define the solid and use them to find the associated integral.

We can perceive the solid in two space with the following diagram:
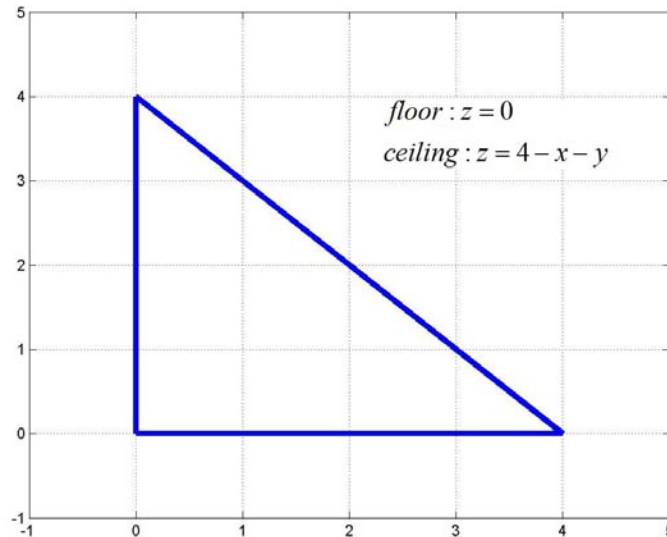
We can then define the solid as the points such that

❖ x goes from $x = 0$ to $x = 4$.
  ➢ For each x, y goes from $y = 0$ to $y = 4 - x$.
   ▪ For each (x, y), z goes from $z = 0$ to $z = 4 - x$

Putting this all together we find that the volume of the solid can be expressed as

$$volume = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} dz\,dy\,dx$$

Example Exercise 10.4.2: Find the volume between the surfaces $z = 1 + x^2 + y^2$ and $z = 9 - x^2 - y^2$.

Solution:

Step 1: Visualize each surface individually and then together in 3 space. If you are unable to visualize a surface well, construct them using cross sections and contours with your 3D kit or use the meshgrid and mesh commands in MATLAB.

$z = 1 + x^2 + y^2$	$z = 9 - x^2 - y^2$	Both Surfaces together
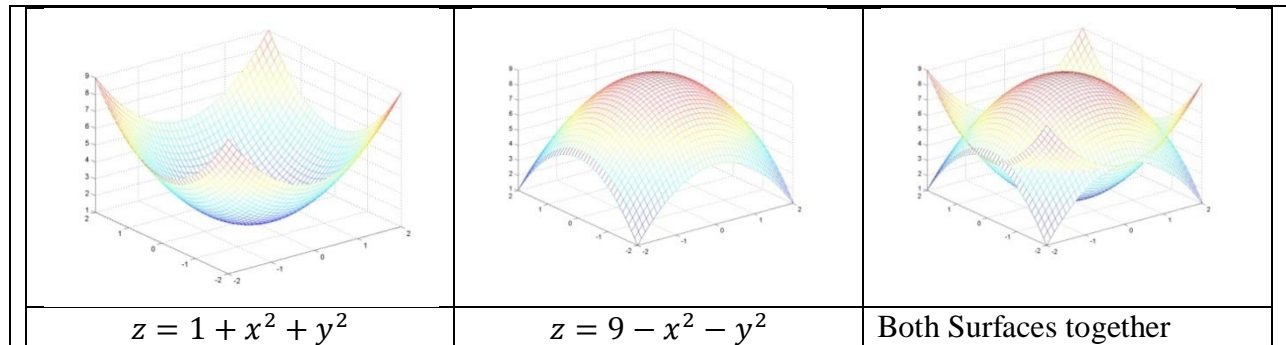
Step 2: Identify the desired solid and find the borders that bound the solid.
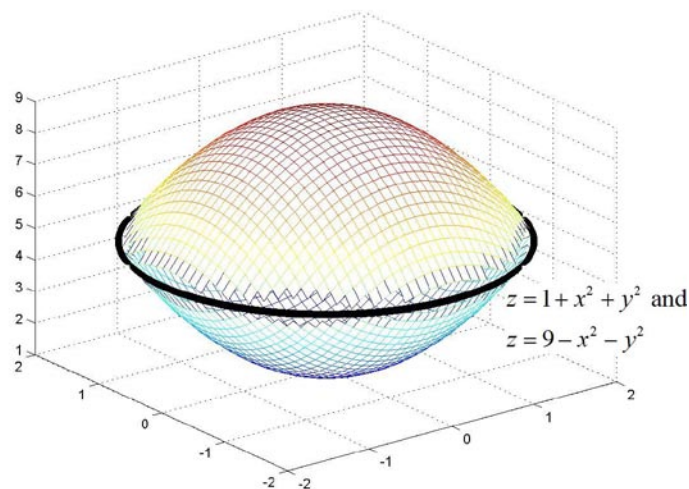
In the above diagram we can see that the solid is bounded below by $z = 1 + x^2 + y^2$ and bounded above by $z = 9 - x^2 - y^2$. The circle shown above represents the x and y values within which the solid will reside. As with example one, this can be thought of as the outermost limit of the wall of the solid and is found where the floor of the solid meets the ceiling. As this circle is found where the floor meets the ceiling, it will satisfy both $z = 1 + x^2 + y^2$ and $z = 9 - x^2 - y^2$. With simple substitution and a little algebra we find that at this border

$$1 + x^2 + y^2 = 9 - x^2 - y^2$$

$$2x^2 + 2y^2 = 8$$

$$x^2 + y^2 = 4$$

Hence the solid is bounded laterally by $x^2 + y^2 = 4$, below by $z = 1 + x^2 + y^2$ and bounded above by $z = 9 - x^2 - y^2$.

Step 3: Define the solid as a region in the xy plane with an associated floor and ceiling.
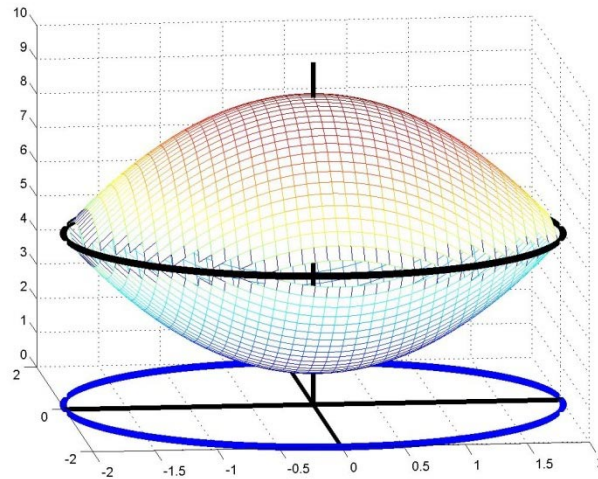
As we can see in the above diagram, the solid can be viewed as the solid above the circle $x^2 + y^2 = 4$ in the xy plane with floor $z = 1 + x^2 + y^2$ and ceiling $z = 9 - x^2 - y^2$.

Step 4: Define the ranges of x, y and z that will define the solid and use them to find the associated integral.

We can perceive the solid in two space with the following diagram:

floor: $z = 1 + x^2 + y^2$
ceiling: $z = 9 - x^2 - y^2$
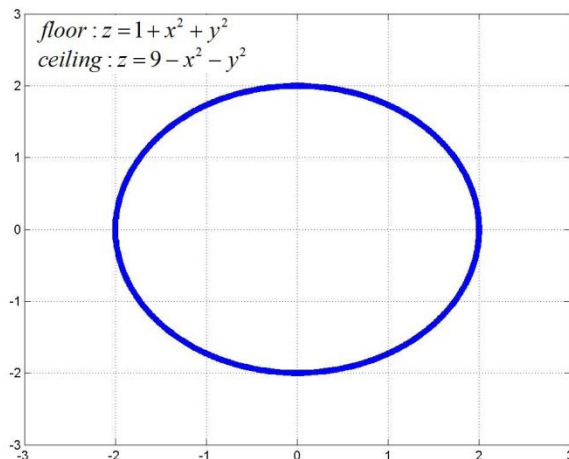
We can then define the solid as the points such that

❖ x goes from $x = -2$ to $x = 2$.
  ➤ For each x, y goes from $y = -\sqrt{4 - x^2}$ to $y = \sqrt{4 - x^2}$.
    ▪ For each (x, y), z goes from $z = 1 + x^2 + y^2$ to $z = 9 - x^2 - y^2$

Putting this all together we find that the volume of the solid can be expressed as

$$volume = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{1+x^2+y^2}^{9-x^2-y^2} dz\,dy\,dx.$$

EXERCISE PROBLEMS:

1) Find the integral that represents the volume above the *xy* plane and below the surface *z = 16 - x² - 4y²* .

2) Find the integral that represents the volume above the surface *z = 3x² + 3y²* and below the surface *z = 16 - x² - y²* .

3) Find the integral that represents the volume above the *xy* plane, below the plane *z = 7 − x - y* and bounded by the planes *x=0* and *y=0* .

4) Find the integral that represents the volume above the surface *z=4x²* , and below the surface *z = 2- 7x²* , and between the planes *y=2* and *y=5*.

5) Find the integral that represents the volume above the surface *z=3y* , and below the surface *z = 18- 8x²* , and bounded by the plane *y = -4*.

6) Find the integral that represents the volume above the surface $z = \sqrt{x^2 + y^2}$ and below the surface $z = \sqrt{18 - 3x^2 - 3y^2}$.

7) A cube $1 \leq x, y, z \leq 3$ (all cm) has density $f(x,y,z) = +$ ($\dfrac{g}{cm^3}$) and we wish to obtain the mass of the cube.

    A. If there are two divisions in x, two divisions in y and two divisions in z and the mass is to be approximated using the minimum value for each variable in each division, find x_1, y_1, z_1 ,y_2 ,z_1 and z_2 use them to fill in the following table with numerical values.

Division	Length	Width	Height	Density	Mass
1					
2					
3					
4					
5					
6					
7					
8					

    B. Fill in the same table below using x_1, y_1, z_1 ,y_2 , z_1 and z_2 ,Δx ,Δy and Δz instead of numerical values. (Note, the divisions should not change between the two tables.)

Division	Length	Width	Height	Density	Mass
1					
2					
3					
4					
5					
6					
7					
8					

    C. Express the approximate mass numerically.
    D. Express the mass obtained in part C in the form $\sum\sum\sum$ $(...)\Delta z \Delta y \Delta x$.
    E. Take the appropriate limits to convert the sum in part D to an integral and evaluate the integral to obtain the precise mass

8) A cubic bird cage with $15 \times 8 \times 10$ (all measured in meters) has density function f with inputs x, y and z (all m) and output $wf(x,y,z)2$ ($\dfrac{birds}{m^3}$). We wish to obtain the number of birds in the cage.

A. If there are two divisions in x, two divisions in y, and two divisions in z and the number of birds is approximated using the middle value for each variable in each division, find x_1, x_2, y_1, z_1 and z_2 use them to fill in the following table with numerical values.

Division	Length	Width	Height	Density	# of birds
1					
2					
3					
4					
5					
6					
7					
8					

B. Use the values of x_1, x_2, y_2 and y_2 to fill in the same table below using x_1, x_2, y_2, z_1 and z_2, Δx, Δy and Δz instead of numerical values. (Note, the divisions should not change between the two tables.)

Division	Length	Width	Height	Density	# of birds
1					
2					
3					
4					
5					
6					
7					
8					

C. Express the approximate number of birds numerically.

D. Express the number of birds obtained in part C in the form $\sum\sum\sum (...)\Delta z \Delta y \Delta x$.

E. Take the appropriate limits to convert the sum in part D to an integral and evaluate the integral to obtain the precise number of birds.